INJECTIVE COLORINGS OF GRAPHS WITH LOW AVERAGE DEGREE

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Abstract. Let $\text{mad}(G)$ denote the maximum average degree (over all subgraphs) of G and let $\chi_i(G)$ denote the injective chromatic number of G. We prove that if $\Delta \geq 4$ and $\text{mad}(G) < \frac{14}{5}$, then $\chi_i(G) \leq \Delta + 2$. When $\Delta = 3$, we show that $\text{mad}(G) < \frac{36}{13}$ implies $\chi_i(G) \leq 5$. In contrast, we give a graph G with $\Delta = 3$, $\text{mad}(G) = \frac{36}{13}$, and $\chi_i(G) = 6$.

1. Introduction

An *injective coloring* of a graph G is an assignment of colors to the vertices of G so that any two vertices with a common neighbor receive distinct colors. The *injective chromatic number*, $\chi_i(G)$, is the minimum number of colors needed for an injective coloring. Injective colorings were introduced by Hahn et al. in [5], and in that paper, the authors showed applications of the injective chromatic number of the hypercube in the theory of error-correcting codes.

Define the *neighboring graph* $G^{(2)}$ by $V(G^{(2)}) = V(G)$ and $E(G^{(2)}) = \{uv : u \text{ and } v \text{ have a common neighbor in } G\}$. Note that $\chi_i(G) = \chi(G^{(2)}) \leq \chi(G^2)$. The chromatic number of G^2 has important applications in Steganography (see [4]).

It is easy to see that $\chi_i(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum degree of G (when the context is clear, we simply write Δ). People are interested in the graphs with relatively small injective chromatic number, and one natural choice of such graphs are planar graphs, or more general, the sparse graphs, see [5, 6, 7]. Let $\text{mad}(G)$ denote the maximum average degree (over all subgraphs) of G. Note that for planar graph G, $\text{mad}(G) < \frac{2g}{g-2}$, where g is the girth of G.

In [2], Doyon, Hahn, and Raspaud showed that for a graph G with maximum degree Δ, the following three results hold: if $\text{mad}(G) < \frac{14}{5}$, then $\chi_i(G) \leq \Delta + 3$; if $\text{mad}(G) < 3$, then $\chi_i(G) \leq \Delta + 4$; and if $\text{mad}(G) < \frac{10}{3}$, then $\chi_i(G) \leq \Delta + 8$.

In [1] the present authors improved some bounds given in [2] and [7] in certain cases; specifically, we studied sufficient conditions to imply $\chi_i(G) = \Delta$ and $\chi_i(G) \leq \Delta + 1$. In the current paper, we study conditions such that $\chi_i(G) \leq \Delta + 2$. Our main result is the following theorem.

Theorem 1. *Let G be a graph with maximum degree $\Delta \geq 4$. If mad$(G) < \frac{14}{5}$, then $\chi_i(G) \leq \Delta + 2$.*

Note that for $\Delta = 3$, we have graphs with $\chi_i(G) = 6$, even with $\text{mad}(G) = \frac{36}{13}$.

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Example 1. *Let G be the incidence graph of the Fano Plane. Observe that G is 3-regular, bipartite, and vertex-transitive. Consider $H = G - v$, where v is an arbitrary vertex. To see that $\chi_i(H) = 6$, we only need to note that the vertices in the part of size 6 form a clique in $H^{(2)}$, but the vertices in the part of size 7 do not.*

We will show that one cannot construct subcubic graphs with $\chi_i(G) = 6$ and $\text{mad}(G) < \frac{36}{13}$.

Theorem 2. *If $\Delta = 3$ and mad$(G) < \frac{36}{13}$, then $\chi_i(G) \leq 5$.*

Hahn, Raspaud, Wang [6] conjectured that every planar graph G with maximum degree Δ has $\chi_i(G) \leq \lceil \frac{3\Delta}{2} \rceil$. For $\Delta = 3$, the conjecture says that $\chi_i(G) \leq 5$. Thus Theorem 2 says that the conjecture is true when the girth of G is at least 8.

The rest of the paper is organized as follows: in Section 2, we introduce the reducible configurations, and as a warmup, we give the proof of Theorem 2; in Section 3, we finish the proof of Theorem 1 by dealing with the cases when $\Delta \geq 6$, $\Delta = 4$, and $\Delta = 5$.

2. Reducible Configurations and Proof of Theorem 2

Before we start, we introduce some notation. A k-vertex is a vertex of degree k; a k^+- and a k^--vertex have degree at least and at most k, respectively. A *thread* is a path with 2-vertices in its interior and 3^+-vertices as its endpoints. A k-thread has k interior 2-vertices. If a 3^+-vertex u is the endpoint of a thread containing a 2-vertex v, then we say that v is a *nearby vertex* of u and vice versa. We write $N_2[u]$ to denote the vertex set consisting of u and its adjacent 2-vertices.

All of our proofs rely on the techniques of reducibility and discharging. We start with a minimal counterexample to the theorem we are proving, and we show that the graph cannot contain certain subgraphs; we call such a subgraph a *reducible configuration*. In the *discharging phase*, we use a counting argument to show that every supposed minimal counterexample must contain a reducible configuration; this yields a contradiction. All of our proofs yield simple algorithms that produce the desired coloring and run in linear time.

Proof of Theorem 2: Assume that G is a minimal counterexample to Theorem 2, that is, G has the specified $\text{mad}(G)$, maximum degree Δ, and $\chi_i(G) > \Delta + 2$. Then the reducible configurations are as follows.

(RC1) G contains no 1-vertices.
(RC2) G contains no 2-threads.
(RC3) G contains no 3-vertex adjacent to two 2-vertices.
(RC4) G contains no adjacent 3-vertices that are each adjacent to a 2-vertex.

Now we show that (RC1) - (RC4) are reducible configurations. In later proofs, when $\Delta > 3$, we will often use the same reducible configurations; so here, we give proofs that do not use the fact $\Delta = 3$, but instead simply assume that every vertex has a list of available colors of size $\Delta + 2$.

(RC1): Let v be a 1-vertex. By the minimality of G, we can color $G - v$ (from its lists). Since v has at most $\Delta - 1$ colors forbidden, we can extend the coloring to G.

(RC2): Let u and v be adjacent 2-vertices. By the minimality of G, we can color $G \setminus \{u, v\}$. Again, we can extend the coloring to G, since each of u and v has at most $(\Delta - 1) + 1$ colors forbidden.

(RC3): Let u be a 3-vertex adjacent to 2-vertices v and w, and let $S = \{u, v, w\}$. By minimality, we can color $G \setminus S$. Note that u has at most $(\Delta - 1) + 1 + 1$ colors forbidden and v and w each have at most $(\Delta - 1) + 1$ colors forbidden. Thus, we can extend the coloring to G.

(RC4): Let u_1 and u_2 be adjacent 3-vertices and v_1 and v_2 be 2-vertices such that v_i is adjacent to u_i, and let $S = \{u_1, u_2, v_1, v_2\}$. By the minimality of G, we can color $G \setminus S$. Note that u_1 and u_2 each have at most $(\Delta - 1) + 1 + 1$ colors forbidden, since the v_is are uncolored. After coloring the u_is, each v_i has at most $(\Delta - 1) + 1 + 1$ colors forbidden. Hence, we can extend the coloring to G.

We use the initial charge $\mu(v) = d(v)$ and the following two discharging rules:

(R1) Each 3-vertex gives charge $\frac{3}{13}$ to each adjacent 2-vertex.
(R2) Each 3-vertex gives charge $\frac{1}{13}$ to each distance-2 2-vertex.

Now we verify that after discharging each vertex has charge at least $\frac{36}{13}$.

Recall that G contains no 1-vertex and observe that (RC2) and (RC3) imply that all vertices that are distance at most two from a 2-vertex must be 3-vertices. Thus, for every 2-vertex v, we have $\mu^*(v) = 2 + 2(\frac{3}{13}) + 4(\frac{1}{13}) = \frac{36}{13}$.

Now we consider 3-vertices. Note that (RC2), (RC3), and (RC4) together imply that a 3-vertex v cannot have 2-vertices at both distance 1 and 2; further, either v has no adjacent 2-vertices and at most three distance-2 2-vertices or else v has at most one adjacent 2-vertex and no distance-2 2-vertices. Hence, we have either $\mu^*(v) \geq 3 - 3(\frac{1}{13}) = \frac{36}{13}$ or $\mu^*(v) \geq 3 - \frac{3}{13} = \frac{36}{13}$.

Thus, the average degree is at least $\frac{36}{13}$. This contradiction completes the proof. □

3. Proof of Theorem 1

To prove Theorem 1, we consider separately the cases $\Delta = 4$, $\Delta = 5$, and $\Delta \geq 6$. The proof when $\Delta \geq 6$ is similar to the proof of Theorem 2, so we consider it first.

Lemma 3. *If $\Delta \geq 6$ and $mad(G) < \frac{14}{5}$, then $\chi_i(G) \leq \Delta + 2$.*

Proof. Below are some reducible configurations.

(RC1) G contains no 1-vertices.
(RC2) G contains no 2-threads.
(RC3) G contains no 3-vertex adjacent to two or three 2-vertices.
(RC4) G contains no 3-vertex adjacent to a 2-vertex and neighbors x and y with $d(x)+d(y) \leq \Delta+2$.
(RC5) G contains no 4-vertex adjacent to four 2-vertices such that one of these 2-vertices has other neighbor with degree less than Δ.

We use the initial charge $\mu(v) = d(v)$ and the following discharging rules.

(R1) each 3^+-vertex gives $\frac{2}{5}$ to each adjacent 2-vertex.
(R2) each vertex with degree at least $\lceil \frac{\Delta+3}{2} \rceil$ gives charge $\frac{2}{5}$ to each adjacent 3-vertex or 4-vertex.
(R3) Suppose that vertex v is adjacent to k 2-vertices and, after applying rules (R1) and (R2), vertex v has charge $\frac{14}{5} + l$ (where $l > 0$). For each adjacent 2-vertex u, vertex v gives charge $\frac{l}{k}$ to the other neighbor of u.

First observe that after applying rules (R1) and (R2), a vertex v has excess charge at least $d(v) - \frac{2}{5}d(v) - \frac{14}{5}$; so each vertex u that receives charge from a vertex v by (R3) receives (from v) a charge of at least $\frac{3}{5} - \frac{14}{5d(v)}$.

Now we verify that all vertices have charge at least $\frac{14}{5}$.

2-vertex: $\mu^*(v) \geq 2 + 2(\frac{2}{5}) = \frac{14}{5}$.

3-vertex: Note that by (RC3) vertex v is adjacent to at most one 2-vertex. If v is adjacent to zero 2-vertices, then $\mu^*(v) = \mu(v) = 3$. If v is adjacent to one 2-vertex, then by (RC4) v also has some neighbor with degree at least $\lceil \frac{\Delta+3}{2} \rceil$. So by rule (R2), $\mu^*(v) \geq 3 - \frac{2}{5} + \frac{2}{5} = 3$.

4-vertex: If v is adjacent to at most three 2-vertices, then $\mu^*(v) \geq 4 - 3(\frac{2}{5}) = \frac{14}{5}$. If v is adjacent to four 2-vertices, then by (RC5), the other neighbor of each adjacent 2-vertex must be a Δ-vertex. Hence, $\mu^*(v) \geq 4 - 4(\frac{2}{5}) + 4(\frac{3}{5} - \frac{14}{5(6)}) > \frac{14}{5}$.

5^+-vertex: $\mu^*(v) \geq d(v) - \frac{2}{5}d(v) = \frac{3}{5}d(v) \geq 3$. □

Now we consider the cases when $\Delta \in \{4, 5\}$. We will need the following two results in our proofs.

Lemma A (Vizing [8]). *For a connected graph G, let L be a list assignment such that $|L(v)| \geq d(v)$ for all v. (a) If $|L(y)| > d(y)$ for some vertex y, then G is L-colorable. (b) If G is 2-connected and the lists are not all identical, then G is L-colorable.*

A graph is *degree-choosable* if it can be colored from its list assignment L whenever $|L(v)| = d(v)$ for every vertex v.

Theorem B (Erdős-Rubin-Taylor [3]). *A graph G fails to be degree-choosable if and only if every block is a complete graph or an odd cycle.*

Lemma 4. *If $\Delta(G) = 4$ and $mad(G) < \frac{14}{5}$, then $\chi_i(G) \leq 6$.*

Proof. Suppose the lemma is false; let G be a minimal counterexample. Below we list some reducible configurations.

(RC1) G contains no 1-vertices.
(RC2) G contains no 2-threads.
(RC3) G contains no 3-vertex adjacent to two or three 2-vertices.
(RC4) G contains no 3-vertex adjacent to one 2-vertex and two 3-vertices.
(RC5) G contains no adjacent 3-vertices with each 3-vertex also adjacent to a (possibly distinct) 2-vertex.

In the first discharging phase, we apply the following two discharging rules:

(R1.1) Every 3^+-vertex gives $\frac{2}{5}$ to each adjacent 2-vertex.
(R1.2) If u is a 3-vertex adjacent to a 4-vertex v and a 2-vertex, then v gives $\frac{1}{5}$ to u.

We consider the charges after the first discharging phase.

2-vertex: $\mu^*(v) = 2 + 2(\frac{2}{5}) = \frac{14}{5}$.

3-vertex: If v is adjacent to a 2-vertex, then by (RC4) v is also adjacent to a 4-vertex, so $\mu^*(v) \geq 3 - \frac{2}{5} + \frac{1}{5} = \frac{14}{5}$. Otherwise, $\mu^*(v) = \mu(v) = 3$.

4-vertex: $\mu^*(v) \geq 4 - 4(\frac{2}{5}) = \frac{12}{5}$.

Note that every 2-vertex and 3-vertex has charge at least $\frac{14}{5}$, but 4-vertices can have insufficient charge. We now construct an auxilliary graph H. Graph H will not contain all the vertices of G, but H will contain every vertex of G that has charge less than $\frac{14}{5}$ after the first discharging phase; H will also contain some of the other vertices. If H is acyclic, then we will show how to complete the discharging argument. If we cannot complete the discharging argument, then we will use H to show that G contains a reducible configuration. More specifically, we construct H so that every cycle in H corresponds to an even cycle in G in which each vertex v satisfies $d_{G^{(2)}}(v) \leq 6$; we show if we cannot complete the discharging argument, then one of these even cycles in G is contained in a reducible configuration.

For convenience, we introduce a subgraph $\widehat{G}^{(2)}$ of $G^{(2)}$. We form $\widehat{G}^{(2)}$ from $G^{(2)}$ by deleting all 2-vertices of G that have degree at most 5 in $G^{(2)}$; we can greedily color these vertices after all others. Hence, it suffices to properly color $\widehat{G}^{(2)}$. We denote the degree of a vertex v in $\widehat{G}^{(2)}$ by $\widehat{d}(v)$. We construct H by the three following rules. We apply rule 3 *after* applying rules 1 and 2 everywhere that they are applicable.

- (H1) If u is a 2-vertex adjacent to vertices v and w, then $v, w \in V(H)$ and $vw \in E(H)$.
- (H2) If u is a 3-vertex adjacent to a 3-vertex v and also adjacent to a 2-vertex, then $u, v \in V(H)$ and $uv \in E(H)$.
- (H3) If $v \in V(H)$ and $\widehat{d}(v) \geq 7$, then for each vertex u adjacent to v in H we create a new vertex v_u in H that is adjacent only to vertex u; finally, we delete vertex v. (We will show that this rule can only apply when $d_G(v) = 4$ and $d_H(v) = 2$.)

Now we have a second discharging phase, with the following three rules:

- (R2.1) Each vertex of degree 1 in H gives a charge of $\frac{1}{5}$ to the bank. (So, if v was replaced by two vertices, v_u and v_w, by rule (H3), then v gives a charge of $\frac{2}{5}$ to the bank.)
- (R2.2) If a vertex v is in H and in G vertex v is adjacent to three vertices of degree 2 and a vertex of degree 3, then the bank gives v a charge of $\frac{1}{5}$.
- (R2.3) If a vertex v is in H and in G vertex v is adjacent to four vertices of degree 2, then the bank gives v a charge of $\frac{2}{5}$.

Let $V_{2,2,2,3}$ denote the number of 4-vertices in G that are adjacent to three vertices of degree 2 and one vertex of degree 3; similarly, let $V_{2,2,2,2}$ denote the number of 4-vertices in G that are adjacent to four vertices of degree 2. Let *Leaves* denote the number of leaves in H. At the end of the second discharging phase, the bank has a charge equal to $\frac{1}{5}(\text{Leaves} - V_{2,2,2,3} - 2V_{2,2,2,2})$; we call this charge the *surplus*. We will show that if the surplus is negative, then G contains a reducible configuration and if the surplus is nonnegative, then every vertex of G has charge at least $\frac{14}{5}$ (which contradicts $\text{mad}(G) < \frac{14}{5}$).

First, we assume the surplus is negative. Note that if the surplus is negative, then it must be negative when restricted to some component J of H. Observe that each vertex counted by $V_{2,2,2,3}$ has degree 3 in H and each counted by $V_{2,2,2,2}$ has degree 4 in H. Thus, if the surplus is negative when restricted to J, then J has average degree greater than 2. Hence, J contains a cycle C and at least one vertex u counted by either $V_{2,2,2,3}$ or $V_{2,2,2,2}$. Recall that $N_2[u]$ is the set consisting of vertex u and all adjacent 2-vertices. By the minimality of G, we have an injective 6-coloring of $G \setminus N_2[u]$ (note that $(G \setminus N_2[u])^{(2)} = G^{(2)} \setminus N_2[u]$); equivalently, this is a proper coloring of $G^{(2)} \setminus N_2[u]$.

Let C' be the shortest cycle in G that contains all the vertices of $V(C)$ in the order in which they appear in C; thus, $V(C')$ contains $V(C)$, as well as some additional 2-vertices and possibly 3-vertices. Let K be the subgraph of G consisting of C' and a shortest path from C' to u (including u); if u lies on C', then we also include in K a 2-vertex that is adjacent to u, but that is not responsible for any edge of C. Our proper coloring of $G^{(2)} \setminus N_2[u]$ can naturally be restricted to a proper coloring of $\widehat{G}^{(2)} \setminus N_2[u]$. We will first modify the coloring of $\widehat{G}^{(2)} \setminus N_2[u]$ to get a proper coloring of $\widehat{G}^{(2)} - V(K)$, then show how to extend this coloring to $\widehat{G}^{(2)}$.

If u lies on C', then at most one vertex w of $N_2[u]$ is not in K. Beginning with our coloring of $\widehat{G}^{(2)} \setminus N_2[u]$, we greedily color w, then uncolor the vertices of K; this yields a coloring of $G^{(2)} - V(K)$. We now assume that u does not lie on C'. Observe that C' is an even cycle, and hence $V(C')$ forms two disjoint cycles in $G^{(2)}$; the key observation is that because of (RC5), if C' contains an edge created by (H2), then C' contains two successive such edges, yet C' must not contain three successive such edges, since this would force an instance of (RC4).

Let x denote the vertex of degree 3 in K. We call the component of $K^{(2)}$ that includes x the *first component* and we call the other component of $K^{(2)}$ the *second component*. Note that the path from x to u in G is of even length; this is true for the same reason that C' is an even cycle. Hence, vertex u is in the first component and the vertices in $N_2[u]-u$ are in the second component. Starting from our coloring of $\widehat{G}^{(2)} \setminus N_2[u]$, we uncolor all vertices of the second component. We now greedily color the uncolored vertices of the second component that are not on C' in order of decreasing distance from C' (as we show in the next paragraph, this uses at most 6 colors). Finally, we uncolor the vertices of K in the first component; this yields a coloring of $G^{(2)} - V(K)$.

Let $L(v)$ denote the list of remaining available colors at each vertex v. Rule (H3) implies that $\widehat{d}(v) \leq 6$ for each $v \in V(H)$. Since each vertex v of K has $\widehat{d}(v) \leq 6$ and we are allowed 6 colors for our injective coloring of G, we thus have $|L(v)| \geq d_{K^{(2)}}(v)$ for each vertex v. By Lemma A and Theorem B, to complete the coloring of $\widehat{G}^{(2)}$, it suffices to show that each component of $K^{(2)}$ either contains a vertex w with $|L(w)| > d_{K^{(2)}}(w)$ or contains a block that is neither a clique nor an odd cycle.

Since u is counted by either $V_{2,2,2,3}$ or $V_{2,2,2,2}$, we have $\widehat{d}(u) < 6$; hence, we conclude $d_{K^{(2)}}(u) < |L(u)|$. Thus, we can extend the coloring of $\widehat{G}^{(2)} - V(K)$ to the first component. Clearly, the second component contains a cycle E. Note that the two neighbors of x that lie on E (and are adjacent to each other in E) also have a common neighbor in $K^{(2)}$; hence, the second component contains a block that is not a cycle or a clique. Thus, we can extend the coloring of $\widehat{G}^{(2)} - V(K)$ to the second component. Hence, we have shown that if the surplus is negative, then $\widehat{G}^{(2)}$ contains a reducible configuration.

We now show that if the surplus is nonnegative, then the average degree in G is at least $\frac{14}{5}$. We must verify that after each leaf in H gives a charge of $\frac{1}{5}$ to the bank and each vertex in H counted by $V_{2,2,2,3}$ or $V_{2,2,2,2}$ receives charge from the bank, every vertex has charge at least $\frac{14}{5}$. Note that if $d_G(v) \leq 2$, then $v \notin H$. To denote the charge at each vertex v after the second discharging phase, we write $\mu^{**}(v)$.

First we consider a vertex $v \in V(H)$ such that $d_G(v) = 3$. Suppose that $d_H(v) = 1$. Recall that each 2-vertex that is adjacent to v in G corresponds to an edge incident to v in H. Since $d_H(v) = 1$, v is adjacent in G to at most one 2-vertex. Further, if v is adjacent to a 2-vertex, then v is not adjacent to a 3-vertex (since this would imply $d_H(v) \geq 2$). Hence, either v is adjacent in G to one 2-vertex and two 4-vertices or v is not adjacent in G to any 2-vertices. In each case, $\mu^*(v) = 3$, so v can give charge $\frac{1}{5}$ to the bank, ending with charge $\mu^{**}(v) = 3 - \frac{1}{5} = \frac{14}{5}$.

Now suppose that $d_H(v) \geq 2$. Either v is adjacent in G to a 2-vertex, a 3-vertex, and a 4-vertex, or v is adjacent in G to at least two 3-vertices and to no 2-vertices. In the first case $\mu^*(v) = 3 - 1(\frac{2}{5}) + 1(\frac{1}{5}) = \frac{14}{5}$, and in the second case $\mu^*(v) = \mu(v) = 3$. Note further that in the first case, $d_{G^{(2)}}(v) \leq 6$ and in the second case, $d_{G^{(2)}}(v) \leq 7$. However, in the second case each 3-vertex that is adjacent to v in H is adjacent to a 2-vertex in G that is deleted in $\widehat{G}$; so we have $\widehat{d}(v) \leq 5$. Hence, in each case $\widehat{d}(v) \leq 6$, so rule (H3) never applies to a vertex $v \in V(H)$ such that $d_G(v) = 3$. Thus, in both cases we have $\mu^{**}(v) = \mu^*(v) \geq \frac{14}{5}$.

Now we consider a vertex $v \in V(H)$ such that $d_G(v) = 4$. If vertex v is adjacent in G to at least three 2-vertices, then $\widehat{d}(v) \leq 6$, so rule (H3) does not apply to v. Hence, if v is counted by $V_{2,2,2,3}$, then $\mu^*(v) \geq 4 - 3(\frac{2}{5}) - 1(\frac{1}{5}) = \frac{13}{5}$ and $\mu^{**}(v) = \mu^*(v) + \frac{1}{5} = \frac{14}{5}$; similarly, if v is counted by $V_{2,2,2,2}$, then $\mu^*(v) = 4 - 4(\frac{2}{5}) = \frac{12}{5}$ and $\mu^{**}(v) = \mu^*(v) + \frac{2}{5} = \frac{14}{5}$. If during the initial discharging phase, v only gave charge to two 2-vertices (and no 3-vertices), then v has sufficient charge to give to the bank if it is split by rule (H3): $\mu^{**}(v) \geq \mu^*(v) - 2(\frac{1}{5}) = 4 - 2(\frac{2}{5}) - 2(\frac{1}{5}) = \frac{14}{5}$. Hence, we need only consider the case when during the first discharging phase v gave charge to at most two 2-vertices and at least one 3-vertex. We examine three subcases.

If v is adjacent in G to two 2-vertices and two 3-vertices, then $\widehat{d}(v) \leq 6$, so rule (H3) does not apply to v; hence $\mu^{**}(v) = \mu^*(v) = 4 - 2(\frac{2}{5}) - 2(\frac{1}{5}) = \frac{14}{5}$. If v is adjacent to at most one 2-vertex, then after the initial discharging phase, $\mu^*(v) \geq 4 - \frac{2}{5} - 3(\frac{1}{5}) = 3$, so $\mu^{**}(v) = \mu^*(v) - \frac{1}{5} = \frac{14}{5}$. Finally, suppose that v gave charge to two 2-vertices and one 3-vertex. If the final neighbor of v is a 4-vertex, then $d_{G^{(2)}}(v) = 7$. However, the 3-vertex adjacent to v is also adjacent to a 2-vertex u. Because $d_{G^{(2)}}(u) \leq 5$, we have $\widehat{d}(v) \leq 6$, so rule (H3) does not apply to v. Hence $\mu^{**}(v) = \mu^*(v) = 4 - 2(\frac{2}{5}) - 1(\frac{1}{5}) = 3$. □

The proof of Lemma 5 is similar to the proof of Lemma 4, but slightly more complicated. The additional obstacle we must address in the current proof is verifying that each 5-vertex has sufficient charge. The additional asset we have is that we are allowed to use 7 colors (rather than the 6 colors allowed in Lemma 4).

Lemma 5. *If $\Delta(G) = 5$ and $mad(G) < \frac{14}{5}$, then $\chi_i(G) \leq 7$.*

Proof. Suppose the lemma is false; let G be a minimal counterexample. Below are some reducible configurations.

(RC1) G contains no 1-vertices.
(RC2) G contains no 2-threads.
(RC3) G contains no 3-vertex adjacent to two or three 2-vertices.
(RC4) G contains no 3-vertex adjacent to one 2-vertex and two other vertices u and v with $d(u) + d(v) \leq 7$.

In the first discharging phase, we apply the following three discharging rules:

(R1.1) Every 3^+-vertex gives $\frac{2}{5}$ to each adjacent 2-vertex.

(R1.2) If u is a 3-vertex adjacent to two 4-vertices and a 2-vertex, then each adjacent 4-vertex gives $\frac{1}{5}$ to u.

(R1.3) Every 5-vertex gives $\frac{2}{5}$ to each adjacent 3-vertex that is adjacent to a 2-vertex and gives $\frac{1}{5}$ to each adjacent 4-vertex.

We consider the charges after the first discharging phase.

2-vertex: $\mu^*(v) = 2 + 2(\frac{2}{5}) = \frac{14}{5}$.

3-vertex: If v is adjacent to a 2-vertex, then by (RC4) v is either adjacent to two 4-vertices or adjacent to a 5-vertex. In the first case, $\mu^*(v) = 3 - \frac{2}{5} + 2(\frac{1}{5}) = 3$. In the second case, $\mu^*(v) = 3 - \frac{2}{5} + \frac{2}{5} = 3$. Otherwise, $\mu^*(v) = \mu(v) = 3$.

4-vertex: $\mu^*(v) \geq 4 - 4(\frac{2}{5}) = \frac{12}{5}$.

5-vertex: $\mu^*(v) \geq 5 - 5(\frac{2}{5}) = 3$.

For convenience, we introduce a subgraph $\tilde{G}^{(2)}$ of $G^{(2)}$. We form $\tilde{G}^{(2)}$ from $G^{(2)}$ by deleting all vertices of G that have degree at most 6 in $G^{(2)}$; we can greedily color these vertices after all others. We denote the degree of a vertex v in $\tilde{G}^{(2)}$ by $\tilde{d}(v)$. (Note the subtle difference from the proof of Lemma 4: to form $\widehat{G}^{(2)}$ we only deleted 2-vertices, but now we delete all vertices with $\tilde{d}(v) \leq 6$. This change is necessary to accomodate the 5-vertices.) Hence, it suffices to properly color $\tilde{G}^{(2)}$. Again we construct an auxiliary graph H, to help finish the discharging argument. We construct H by the two following rules:

(H1) If u is a 2-vertex adjacent to a 4-vertex v and also adjacent to w, then $v, w \in V(H)$ and $vw \in E(H)$.

(H2) If $v \in V(H)$ and $\tilde{d}(v) \geq 8$, then we split v into multiple copies in H, as follows. For each edge e incident to v in H, we create a new vertex v_e that is incident only to edge e, then we delete the original copy of v in H.

Now we have a second discharging phase, with the following four rules:

(R2.1) Each vertex of degree 1 in H gives a charge of $\frac{1}{5}$ to the bank. (So, if v was split into k vertices by rule (H2), then v gives a charge of $\frac{k}{5}$ to the bank.)

(R2.2) If a vertex v is in H and in G vertex v is adjacent to three vertices of degree 2 and a vertex of degree 3, then the bank gives v a charge of $\frac{1}{5}$.

(R2.3) If a vertex $v \in V(H)$ and in G vertex v is adjacent to four vertices of degree 2, then the bank gives v a charge of $\frac{2}{5}$.

(R2.4) If a 4-vertex v has charge at least 3 after applying rules (R2.1), (R2.2), (R2.3), then v sends charge $\frac{1}{15}$ to each 5-vertex v at distance 2 that has a common neighbor w with u such that $d_G(w) = 2$.

Let $V_{2,2,2,3}$ denote the number of 4-vertices in G that are adjacent to three vertices of degree 2 and one vertex of degree 3; similarly, let $V_{2,2,2,2}$ denote the number of 4-vertices in G that are adjacent to four vertices of degree 2. Let *Leaves* denote the number of leaves in H. At the end of the second discharging phase, the bank has a surplus equal to $\frac{1}{5}(\text{Leaves} - V_{2,2,2,3} - 2V_{2,2,2,2})$. We will show that if the surplus is negative, then G contains a reducible configuration and if the surplus is nonnegative, then every vertex of G has charge at least $\frac{14}{5}$ (which contradicts $\text{mad}(G) < \frac{14}{5}$).

First, we assume the surplus is negative. Note that if the surplus is negative, then it must be negative when restricted to some component J of H. Observe that each vertex counted by $V_{2,2,2,3}$ has degree 3 in H and each counted by $V_{2,2,2,2}$ has degree 4 in H. Thus, if the surplus is negative when restricted to J, then J has average degree greater than 2. Hence, J contains a cycle C and at least one vertex u counted by either $V_{2,2,2,3}$ or $V_{2,2,2,2}$. Recall that $N_2[u]$ is the set consisting of vertex u and all adjacent 2-vertices. By the minimality of G, we have an injective 7-coloring of $G \setminus N_2[u]$ (note that $(G \setminus N_2[u])^{(2)} = G^{(2)} \setminus N_2[u]$); equivalently, this is a proper coloring of $G^{(2)} \setminus N_2[u]$.

Let C' be the shortest cycle in G that contains all the vertices of $V(C)$ in the order in which they appear in C; thus, $V(C')$ contains $V(C)$, as well as some additional 2-vertices. Let K be the subgraph of G consisting of C' and a shortest path from C' to u (including u); if u lies on C', then we also include in K a 2-vertex that is adjacent to u, but that is not responsible for any edge of C. Our proper coloring of $G^{(2)} \setminus N_2[u]$ can naturally be restricted to a proper coloring of $\tilde{G}^{(2)} \setminus N_2[u]$. We will first modify the coloring of $\tilde{G}^{(2)} \setminus N_2[u]$ to get a proper coloring of $\tilde{G}^{(2)} - V(K)$, then show how to extend this coloring to $\tilde{G}^{(2)}$.

If u lies on C', then at most one vertex w of $N_2[u]$ is not in K. Beginning with our coloring of $\tilde{G}^{(2)} \setminus N_2[u]$, we greedily color w, then uncolor the vertices of K; this yields a coloring of $G^{(2)} - V(K)$.

We now assume that u does not lie on C'. Observe that C' is an even cycle, and hence $V(C')$ forms two disjoint cycles in $G^{(2)}$; this observation follows directly from the fact that each edge of H is constructed by rule (H1).

Let x denote the vertex of degree 3 in K. We call the component of $K^{(2)}$ that includes x the *first component* and we call the other component of $K^{(2)}$ the *second component*. Note that the path from x to u in G is of even length; this is true for the same reason that C' is an even cycle. Hence, vertex u is in the first component and the vertices in $N_2[u] - u$ are in the second component. Starting from our coloring of $\tilde{G}^{(2)} \setminus N_2[u]$, we uncolor all vertices of the second component.

Let $L(v)$ denote the list of remaining available colors at each vertex v. Rule (H2) implies that $\tilde{d}(v) \leq 7$ for each $v \in V(H)$. Since each vertex v of K has $\tilde{d}(v) \leq 7$ and we are allowed 7 colors for our injective coloring of G, we thus have $|L(v)| \geq d_{K^{(2)}}(v)$ for each vertex v. By Lemma A and Theorem B, to complete the coloring of $\tilde{G}^{(2)}$, it suffices to show that each component of $K^{(2)}$ either contains a vertex w with $|L(w)| > d_{K^{(2)}}(w)$ or contains a block that is neither a clique nor an odd cycle.

Since u is counted by either $V_{2,2,2,3}$ or $V_{2,2,2,2}$, we have $\tilde{d}(u) < 7$; hence, we conclude $d_{K^{(2)}}(u) < |L(u)|$. Thus, we can extend the coloring of $\tilde{G}^{(2)} - V(K)$ to the first component. Clearly, the second component contains a cycle E. Note that the two neighbors of x that lie on E (and are adjacent to each other in E) also have a common neighbor in $K^{(2)}$; hence, the second component contains a block that is not a cycle or a clique. Thus, we can extend the coloring of $\tilde{G}^{(2)} - V(K)$ to the second component. Hence, we have shown that if the surplus is negative, then $\tilde{G}^{(2)}$ contains a reducible configuration.

We now show that if the surplus is nonnegative, then the average degree in G is at least $\frac{14}{5}$. We must verify that after each leaf in H gives a charge of $\frac{1}{5}$ to the bank and each vertex in H counted by $V_{2,2,2,3}$ or $V_{2,2,2,2}$ receives charge from the bank, every vertex has charge at least $\frac{14}{5}$. To denote the charge at each vertex v after the second discharging phase, we write $\mu^{**}(v)$.

First, we consider a vertex $v \in V(H)$ such that $d_G(v) = 3$. Note that $d_H(v) \leq 1$, since $d_H(v) \geq 2$ would imply that in G vertex v is adjacent to at least two 2-vertices, which contradicts (RC3). So suppose that $d_H(v) = 1$. Clearly, v is adjacent to a 2-vertex in G. If v is also adjacent to a 5-vertex, then $\mu^*(v) \geq 3 - \frac{2}{5} + \frac{2}{5} = 3$. If v is not adjacent to a 5-vertex, then by (RC3) and (RC4), v must be adjacent to two 4-vertices; hence, $\mu^*(v) \geq 3 - \frac{2}{5} + 2(\frac{1}{5}) = 3$. In each case, v has charge at least 3 after the initial discharging phase, so v can give charge $\frac{1}{5}$ to the bank.

Now, we consider a vertex $v \in V(H)$ such that $d_G(v) = 4$. We must verify that for each such vertex, either $\tilde{d}(v) \leq 6$ or v is able to give sufficient charge to the bank after it is split by rule (H2). If in G vertex v is adjacent to at least three 2-vertices, then $\tilde{d}(v) \leq 7$. If in the initial discharging phase, v has only given charge to two 2-vertices (and no 3-vertices), then v has sufficient charge to give to the bank if it is split by rule (H2). Hence, we need only consider the case when during the first discharging phase v has given charge to at most two 2-vertices and at least one 3-vertex. Note, as follows, that rule (R2.4) will never cause the charge of a 4-vertex v to drop below $\frac{14}{5}$. If a 4-vertex gives charge by rule (R2.4) to at most three 5-vertices, then $\mu^{**}(v) \geq 3 - 3(\frac{1}{15}) = \frac{14}{5}$. However, if v gives charge by rule (R2.4) to four 5-vertices, then $\mu^{**}(v) = \mu^*(v) - 4(\frac{1}{15}) + \frac{2}{5} > \frac{14}{5}$. Hence, in what follows, we do not consider rule (R2.4). We examine three subcases.

If v is adjacent in G to two 2-vertices and two 3-vertices, then $\tilde{d}(v) \leq 6$. If v is adjacent to at most one 2-vertex, then after the initial discharging phase, v has charge at least $4 - \frac{2}{5} - 3(\frac{1}{5}) = 3$, so v is able to give charge $\frac{1}{5}$ to the bank. Finally, suppose that v has given charge to two 2-vertices and one 3-vertex. Observe that the 3-vertex adjacent to v is also adjacent to a 2-vertex u. Because $d_{G^{(2)}}(u) \leq 6$, we see that $\tilde{d}(v) \leq 7$.

Finally, we consider a vertex $v \in H$ such that $d_G(v) = 5$. If v is adjacent in G to at most three 2-vertices and at most four 3^--vertices, then $\mu^{**}(v) \geq \mu^*(v) - 3(\frac{1}{5}) \geq 5 - 4(\frac{2}{5}) - 3(\frac{1}{5}) = \frac{14}{5}$. Suppose instead that v is adjacent to five 3^--vertices. If v is adjacent to at least three 2-vertices, then $\tilde{d}(v) \leq 7$, so v is not split by rule (H2). Thus, $\mu^{**}(v) \geq 5 - 5(\frac{2}{5}) = 3$. If v is adjacent to five 3^--vertices and at least three of them are 3-vertices, then we have the following analysis. If v is not split by rule (H2), then $\mu^{**}(v) \geq 5 - 5(\frac{2}{5}) = 3$; hence, we assume that v is split by (H2), which implies that $\tilde{d}(v) \geq 8$. This inequality implies that at least three 3-vertices that are adjacent to v are not adjacent to 2-vertices (if such a 3-vertex is adjacent to a 2-vertex u, then $d_{G^{(2)}}(u) \leq 6$, so u does not contribute to $\tilde{d}(v)$). Hence, these 3-vertices do not receive charge from v, so we conclude that $\mu^{**}(v) \geq 5 - 2(\frac{2}{5}) - 2(\frac{1}{5}) = \frac{19}{5} > \frac{14}{5}$.

So v must be adjacent to exactly four 3^--vertices, and all of these 3^--vertices are 2-vertices. Consider $d_H(v)$ before we apply rule (H2). Each edge incident in H to v corresponds to a 2-vertex in G that is adjacent to v and is also adjacent to a 4-vertex u. If at least two of these 4-vertices have $d_{G^{(2)}}(u) \leq 6$, then $\tilde{d}(v) \leq 6$, and v is not split by (H2). Suppose one such 4-vertex u has $d_{G^{(2)}}(u) \geq 7$. Either u is adjacent to a most two 2-vertices, or u is adjacent to three 2-vertices and one 5-vertex; in both cases, $\mu^*(u) \geq 3$, so u gives charge $\frac{1}{15}$ to v. Hence, if at least three of these 4-verts have $d_{G^{(2)}} \geq 7$, then v gets charge $\frac{1}{15}$ from each, so $\mu^{**}(v) \geq 5 - 5(\frac{2}{5}) - 2(\frac{1}{5}) + 3\frac{1}{15} = \frac{14}{5}$. $\square$

By combining Lemmas 3, 4, and 5, we prove Theorem 1.

Although we have stated our results only for injective coloring, all of our proofs yield the same bounds for injective list coloring (which is defined analogously).

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